6. Differential Analysis of Fluid Flow

- **Contents**

  - Various kinematic elements of the flow

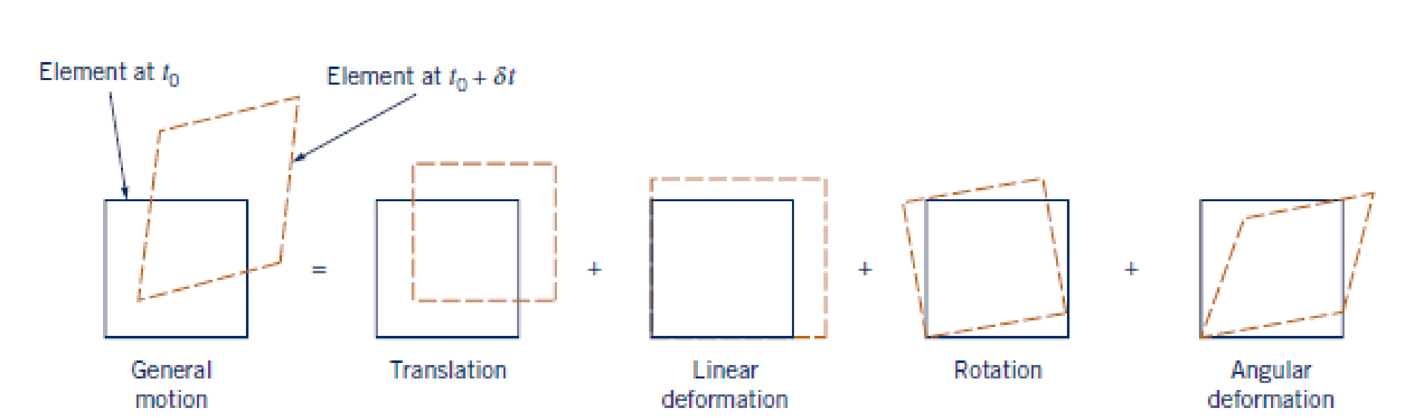
  - The continuity equation

  - Stream function and velocity potential

  - Simple potential flow fields

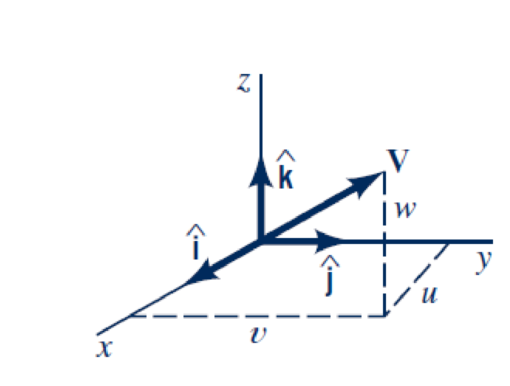
  - Analysis of flow using the Navier-Stokes equations

Fluid Element Kinematics



Fluid element motion consists of translation, linear deformation, rotation, and angular deformation.

Velocity and Acceleration Fields revisited



$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

or $$\mathbf{a} = \frac{D\mathbf{V}}{Dt}$$

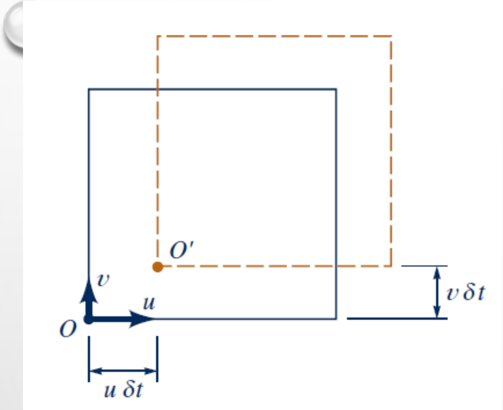
Material derivative, or substantial derivative

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})(\ )$$

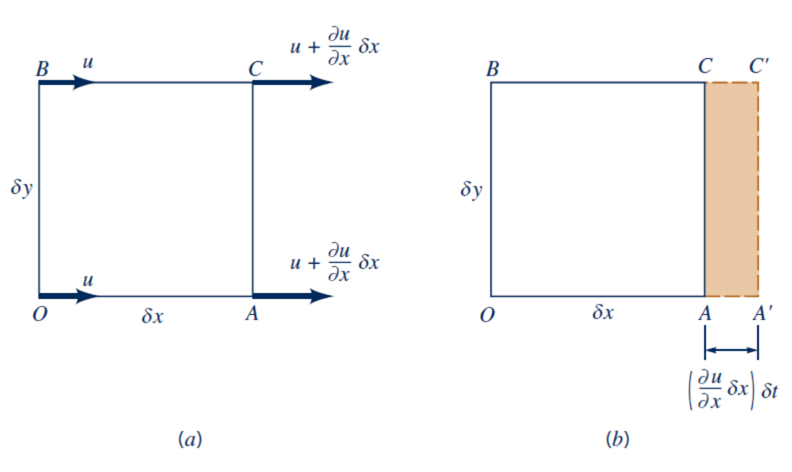
with $$\boldsymbol{\nabla}(\ ) = \frac{\partial(\ )}{\partial x}\hat{\mathbf{i}} + \frac{\partial(\ )}{\partial y}\hat{\mathbf{j}} + \frac{\partial(\ )}{\partial z}\hat{\mathbf{k}}$$

Linear motion and deformation

Translation



Deformation



$$\text{Change in } \delta V\!\!\!\!/ = \left(\frac{\partial u}{\partial x}\delta x\right)(\delta y\, \delta z)(\delta t)$$

The rate of volume change

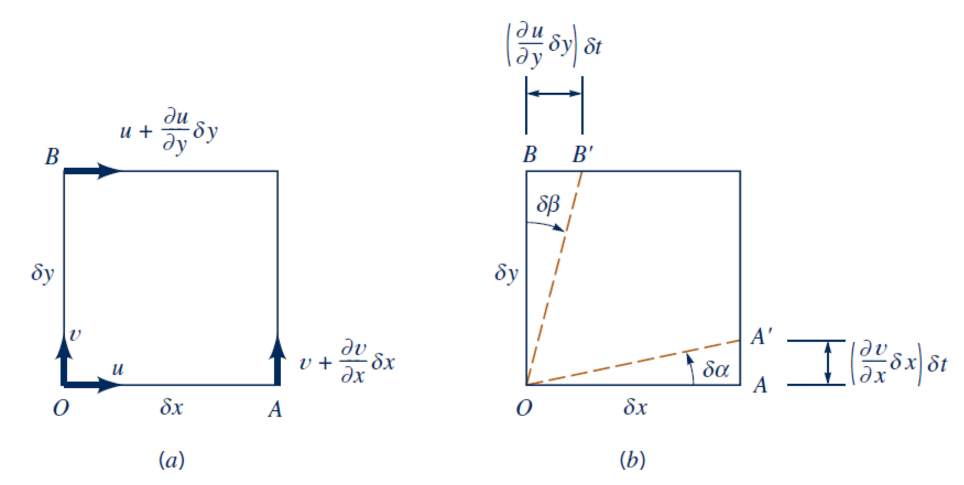
$$\frac{1}{\delta V\!\!\!\!/}\frac{d(\delta V\!\!\!\!/)}{dt} = \lim_{\delta t \to 0}\left[\frac{(\partial u/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

In the general case

$$\frac{1}{\delta V\!\!\!\!/}\frac{d(\delta V\!\!\!\!/)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V}$$

Volumetric dilatation rate

Angular motion and deformation



The angular velocity of line OA

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t}$$

For small angle

$$\tan \delta\alpha \approx \delta\alpha = \frac{(\partial v/\partial x)\,\delta x\,\delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$$

counterclockwise

$$\omega_{OA} = \lim_{\delta t \to 0}\left[\frac{(\partial v/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial v}{\partial x}$$

The angular velocity of line OB

$$\omega_{OB} = \lim_{\delta t \to 0}\left[\frac{(\partial u/\partial y)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial y}$$

clockwise

The rotation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Positive: counterclockwise

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Angular motion and deformation

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{i}} + \omega_y \hat{\mathbf{j}} + \omega_z \hat{\mathbf{k}}$$

$$\boldsymbol{\omega} = \tfrac{1}{2} \operatorname{curl} \mathbf{V} = \tfrac{1}{2} \nabla \times \mathbf{V}$$

$$\frac{1}{2} \nabla \times \mathbf{V} = \frac{1}{2} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}$$

vorticity $\boldsymbol{\zeta} = 2\,\boldsymbol{\omega} = \nabla \times \mathbf{V}$

If $\nabla \times \mathbf{V} = 0$ ➡ irrotational

Angular deformation $\delta\gamma = \delta\alpha + \delta\beta$

Rate of shearing strain or rate of angular deformation

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0} \left[\frac{(\partial v/\partial x)\,\delta t + (\partial u/\partial y)\,\delta t}{\delta t}\right]$$

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

EXAMPLE 6.1 Vorticity

GIVEN For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = (x^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$$

FIND Is this flow irrotational?

SOLUTION

For an irrotational flow the rotation vector, $\boldsymbol{\omega}$, having the components given by Eqs. 6.12, 6.13, and 6.14 must be zero. For the prescribed velocity field

$$u = x^2 - y^2 \qquad v = -2xy \qquad w = 0$$

and therefore

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

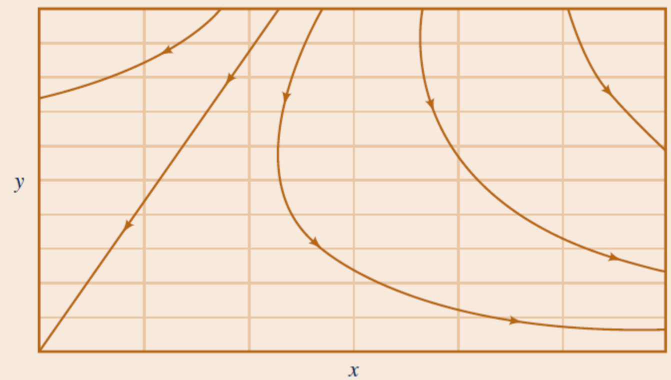
$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[(-2y) - (-2y)] = 0$$

Thus, the flow is irrotational. **(Ans)**

COMMENTS It is to be noted that for a two-dimensional flow field (where the flow is in the x–y plane) ω_x and ω_y will always be zero, since by definition of two-dimensional flow u and v are not functions of z, and w is zero. In this instance the condition for irrotationality simply becomes $\omega_z = 0$ or $\partial v/\partial x = \partial u/\partial y$.

The streamlines for the steady, two-dimensional flow of this example are shown in Fig. E6.1. (Information about how to calculate



■ Figure E6.1

streamlines for a given velocity field is given in Sections 4.1.4 and 6.2.3.) It is noted that all of the streamlines (except for the one through the origin) are curved. However, because the flow is irrotational, there is no rotation of the fluid elements. That is, lines OA and OB of Fig. 6.4 rotate with the same speed but in opposite directions.

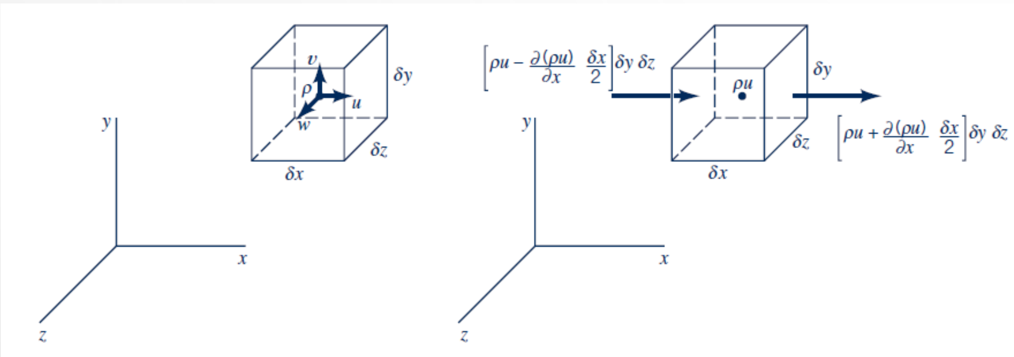
As shown by Eq. 6.17, the condition of irrotationality is equivalent to the fact that the vorticity, ζ, is zero or the curl of the velocity is zero.

Conservation of mass

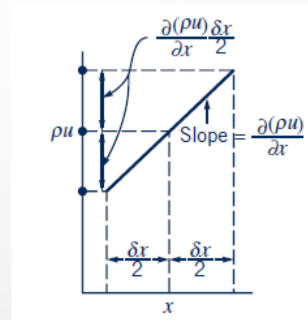
$$\frac{DM_{\text{sys}}}{Dt} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}\int_{\text{cv}} \rho \, d\mathcal{V} + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA = 0$$

Continuity equation

Differential form of continuity equation



(a) (b)



$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho \, d\mathcal{V} \approx \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z$$

$$\rho u|_{x+(\delta x/2)} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}$$

$$\rho u|_{x-(\delta x/2)} = \rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}$$

$$\begin{array}{c}\text{Net rate of mass} \\ \text{outflow in } x \text{ direction}\end{array} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z = \frac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z$$

$$\begin{array}{c}\text{Net rate of mass} \\ \text{outflow in } y \text{ direction}\end{array} = \frac{\partial(\rho v)}{\partial y}\,\delta x\,\delta y\,\delta z$$

$$\begin{array}{c}\text{Net rate of mass} \\ \text{outflow in } z \text{ direction}\end{array} = \frac{\partial(\rho w)}{\partial z}\,\delta x\,\delta y\,\delta z$$

$$\begin{array}{c}\text{Net rate of} \\ \text{mass outflow}\end{array} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z$$

Differential form of continuity equation

The differential equation for conservation of mass / Continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

For steady flow of compressible fluids

$$\nabla \cdot \rho \mathbf{V} = 0$$

or

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

For incompressible fluids

$$\nabla \cdot \mathbf{V} = 0$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

<u>For cylindrical polar coordinates</u>

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

For steady flow of compressible fluids

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

For incompressible fluids

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Example 6.2 Continuity Equation

GIVEN The velocity components for a certain incompressible, steady-flow field are

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z$$
$$w = ?$$

FIND Determine the form of the z component, w, required to satisfy the continuity equation.

Solution

Any physically possible velocity distribution must for an incompressible fluid satisfy conservation of mass as expressed by the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For the given velocity distribution

$$\frac{\partial u}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial v}{\partial y} = x + z$$

so that the required expression for $\partial w/\partial z$ is

$$\frac{\partial w}{\partial z} = -2x - (x + z) = -3x - z$$

Integration with respect to z yields

$$w = -3xz - \frac{z^2}{2} + f(x, y) \qquad \textbf{(Ans)}$$

COMMENT The third velocity component cannot be explicitly determined since the function $f(x, y)$ can have any form and conservation of mass will still be satisfied. The specific form of this function will be governed by the flow field described by these velocity components—that is, some additional information is needed to completely determine w.

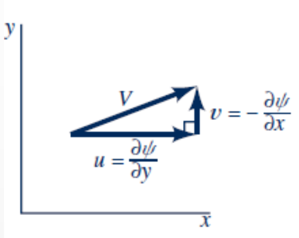
The stream function

For steady, incompressible, plane, 2D flow

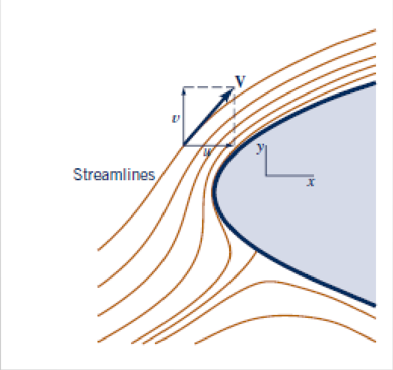
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Stream function ψ (x,y)

$$\boxed{u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}}$$



The continuity equation is identically satisfied



<u>Lines along which ψ is constant are streamlines</u>

The change in the value of ψ

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\, dx + u\, dy$$

Along a line of constant ψ, $d\psi=0$

$$-v\, dx + u\, dy = 0$$

$$\frac{dy}{dx} = \frac{v}{u}$$

Equation for a streamline

The stream function

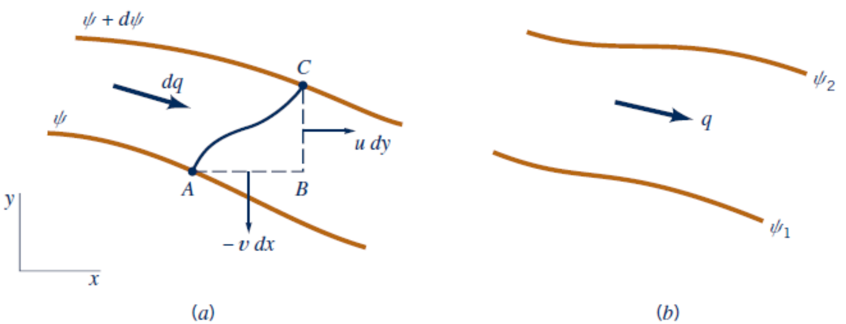


Figure 6.8 The flow between two streamlines.

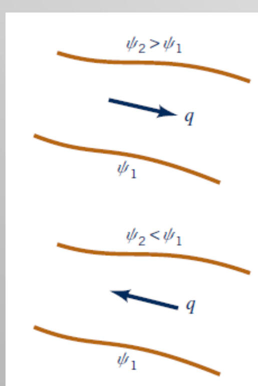
<u>The change in the value of ψ is related to the volume flow rate of flow</u>

dq: the volume rate of flow passing between the two streamlines

$$dq = u\,dy - v\,dx$$

$$dq = \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx$$

$$dq = d\psi$$

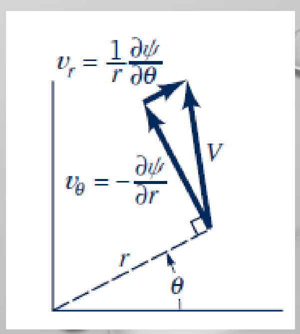


$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

In cylindrical coordinates

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

$$\boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}}$$



EXAMPLE 6.3 Stream Function

GIVEN The velocity components in a steady, incompressible, two-dimensional flow field are

$$u = 2y$$
$$v = 4x$$

FIND

(a) Determine the corresponding stream function and

(b) Show on a sketch several streamlines. Indicate the direction of flow along the streamlines.

SOLUTION

(a) From the definition of the stream function (Eqs. 6.37)

$$u = \frac{\partial \psi}{\partial y} = 2y$$

and

$$v = -\frac{\partial \psi}{\partial x} = 4x$$

The first of these equations can be integrated to give

$$\psi = y^2 + f_1(x)$$

where $f_1(x)$ is an arbitrary function of x. Similarly from the second equation

$$\psi = -2x^2 + f_2(y)$$

where $f_2(y)$ is an arbitrary function of y. It now follows that in order to satisfy both expressions for the stream function

$$\psi = -2x^2 + y^2 + C \qquad \textbf{(Ans)}$$

where C is an arbitrary constant.

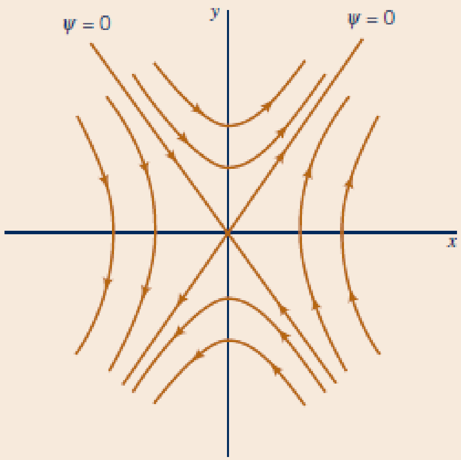
COMMENT Since the velocities are related to the derivatives of the stream function, an arbitrary constant can always be added to the function, and the value of the constant is actually of no consequence. Usually, for simplicity, we set $C = 0$ so that for this particular example the simplest form for the stream function is

$$\psi = -2x^2 + y^2 \qquad (1) \qquad \textbf{(Ans)}$$

Either answer indicated would be acceptable.

(b) Streamlines can now be determined by setting ψ = constant and plotting the resulting curve. With the preceding expression for ψ (with $C = 0$) the value of ψ at the origin is zero, so that the equation of the streamline passing through the origin (the $\psi = 0$ streamline) is

$$0 = -2x^2 + y^2$$



■ Figure E6.3

or

$$y = \pm\sqrt{2}x$$

Other streamlines can be obtained by setting ψ equal to various constants. It follows from Eq. 1 that the equations of these streamlines (for $\psi \neq 0$) can be expressed in the form

$$\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1$$

which we recognize as the equation of a hyperbola. Thus, the streamlines are a family of hyperbolas with the $\psi = 0$ streamlines as asymptotes. Several of the streamlines are plotted in Fig. E6.3. Since the velocities can be calculated at any point, the direction of flow along a given streamline can be easily deduced. For example, $v = -\partial\psi/\partial x = 4x$ so that $v > 0$ if $x > 0$ and $v < 0$ if $x < 0$. The direction of flow is indicated on the figure.

Conservation of Linear Momentum

The linear momentum equation

$$\mathbf{F} = \left.\frac{D\mathbf{P}}{Dt}\right|_{\text{sys}}$$ where $$\mathbf{P} = \int_{\text{sys}} \mathbf{V}\, dm$$

In Chap. 5

$$\sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}} = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho\, d\mathcal{V} + \int_{\text{cs}} \mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

$$\delta\mathbf{F} = \frac{D(\mathbf{V}\,\delta m)}{Dt}$$

$$\delta\mathbf{F} = \delta m\,\frac{D\mathbf{V}}{Dt}$$

$$\delta\mathbf{F} = \delta m\,\mathbf{a}$$

Description of Forces Acting on the Differential Element

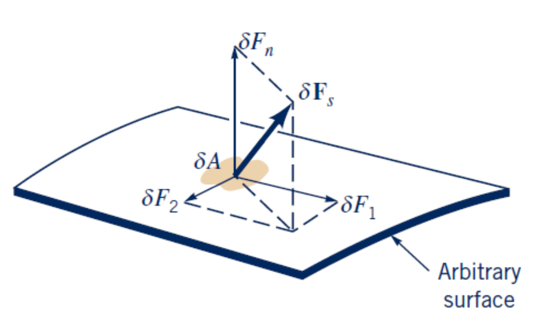
Surface forces vs. body forces

Example of <u>body force</u>

$$\delta \mathbf{F}_b = \delta m\, \mathbf{g}$$

$$\delta F_{bx} = \delta m\, g_x$$
$$\delta F_{by} = \delta m\, g_y$$
$$\delta F_{bz} = \delta m\, g_z$$

<u>Surface force</u>

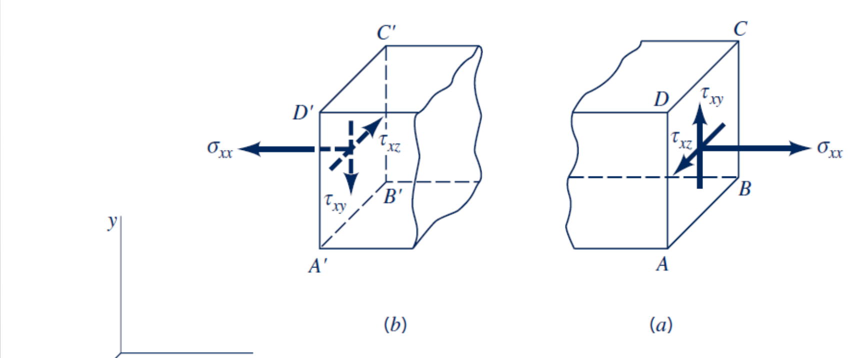


Normal stress

$$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$

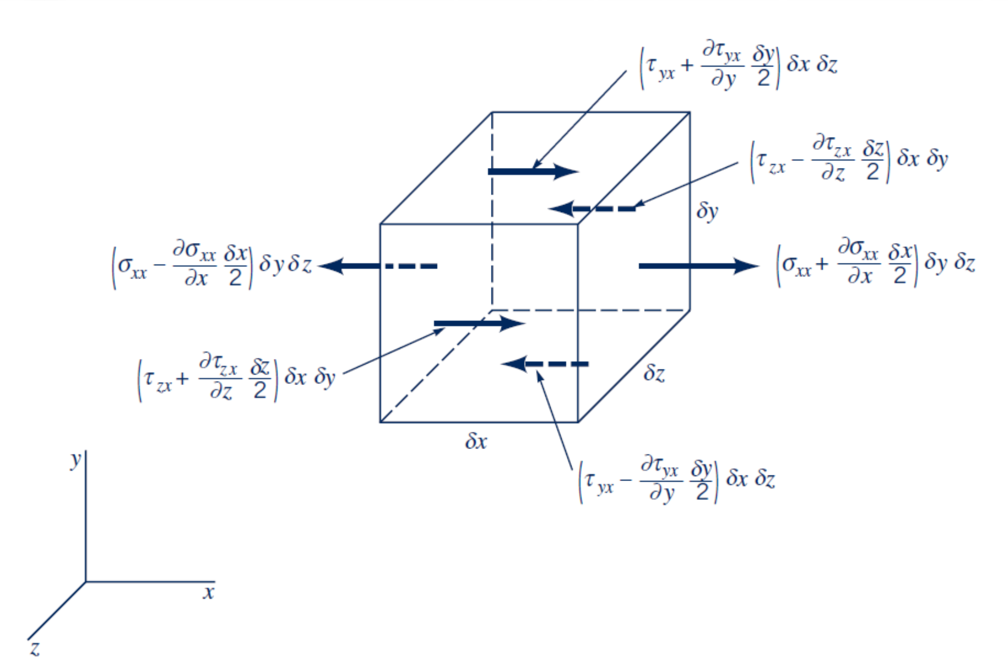
Shearing stresses

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$



Sign convention

Description of Forces Acting on the Differential Element



Surface forces in the x direction

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

In the y and z directions

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

$$\delta \mathbf{F}_s = \delta F_{sx}\hat{\mathbf{i}} + \delta F_{sy}\hat{\mathbf{j}} + \delta F_{sz}\hat{\mathbf{k}}$$

The total resultant force

$$\delta \mathbf{F} = \delta \mathbf{F}_s + \delta \mathbf{F}_b$$

Equations of Motion

$$\delta F_x = \delta m\, a_x$$

$$\delta F_y = \delta m\, a_y$$

$$\delta F_z = \delta m\, a_z$$

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

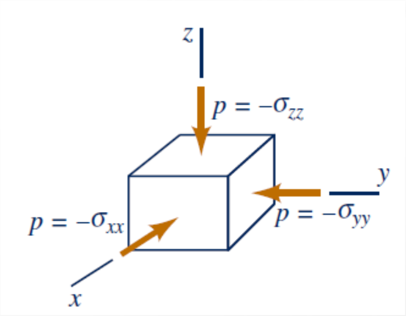
$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Inviscid Flow

Inviscid, nonviscous, or frictionaless

➔ No shearing stresses / The normal stress at a point is independent of direction

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$



Euler's Equation of Motion

Zero shearing stress / the normal stresses: -p

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

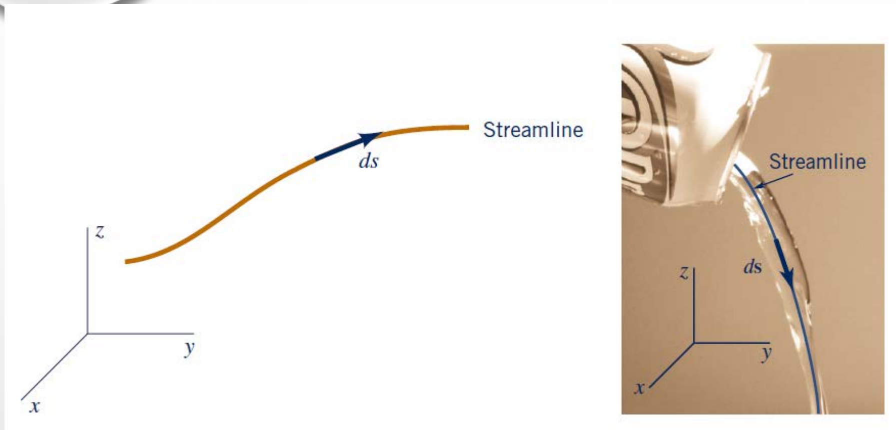
$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

or

$$\rho \mathbf{g} - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]$$

Euler's equation of motion

The Bernoulli Equation



Euler's equation

$$\rho\mathbf{g} - \nabla p = \rho(\mathbf{V}\cdot\nabla)\mathbf{V}$$

Acceleration of gravity vector

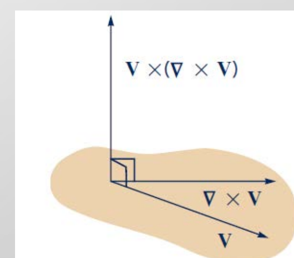
$$\mathbf{g} = -g\nabla z$$

Vector identity $$(\mathbf{V}\cdot\nabla)\mathbf{V} = \tfrac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V})$$

$$\rho\mathbf{g} - \nabla p = \rho(\mathbf{V}\cdot\nabla)\mathbf{V} \;\Rightarrow\; -\rho g\nabla z - \nabla p = \frac{\rho}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \rho\mathbf{V}\times(\nabla\times\mathbf{V})$$

or $$\frac{\nabla p}{\rho} + \frac{1}{2}\nabla(V^2) + g\nabla z = \mathbf{V}\times(\nabla\times\mathbf{V})$$

The dot product with a differential length ds along a streamline



0

$$\frac{\nabla p}{\rho}\cdot d\mathbf{s} + \frac{1}{2}\nabla(V^2)\cdot d\mathbf{s} + g\nabla z\cdot d\mathbf{s} = [\mathbf{V}\times(\nabla\times\mathbf{V})]\cdot d\mathbf{s}$$

$$\Rightarrow\; \frac{dp}{\rho} + \frac{1}{2}d(V^2) + g\,dz = 0$$ or $$\int\frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

The Bernoulli Equation

For inviscid, incompressible fluids (ideal fluids)

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}$$

Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

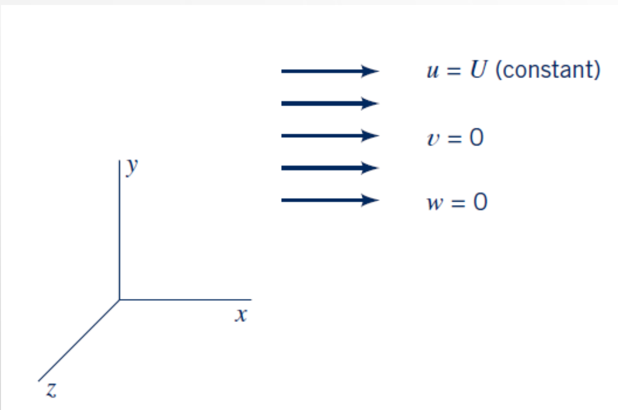
It is applicable under the following restrictions

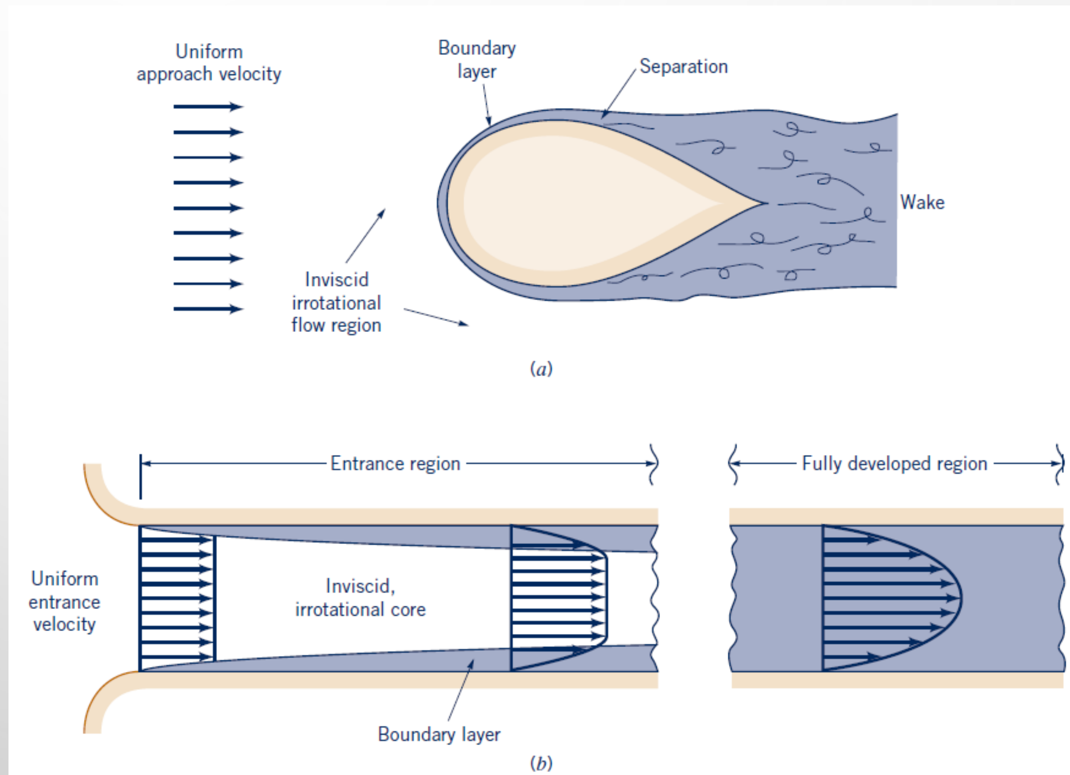
- inviscid flow
- incompressible flow
- steady flow
- flow along a streamline

Irrotational Flow

Irrotational flow field $\nabla \times \mathbf{V} = 0$

i.e., $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$ ➡ $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$ $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$





The Bernoulli Equation for Irrotational Flow

Irrotational flow field $\nabla \times \mathbf{V} = 0$

$$\Rightarrow \quad [\mathbf{V} \times (\nabla \times \mathbf{V})] \cdot d\mathbf{s} = 0$$

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

For incompressible, irrotational flow

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2}$$

Between any two points in the flow field

It is restricted to

- inviscid flow
- steady flow
- incompressible flow
- irrotational flow

The Velocity Potential

For <u>irrotational flow</u>, the velocity component can be expressed using a scalar function $\phi(x,y,z,t)$

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

ϕ: velocity potential

or $\boxed{\mathbf{V} = \nabla \phi}$

Conservation of mass for an incompressible fluid $\nabla \cdot \mathbf{V} = 0$

$\Rightarrow$ $\nabla^2 \phi = 0$ or $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$ In Cartesian coordinate

Laplace equation

In cylindrical coordinate

$$\nabla \phi = \frac{\partial \phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial \phi}{\partial z}\hat{\mathbf{e}}_z$$

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad v_z = \frac{\partial \phi}{\partial z}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

EXAMPLE 6.4 Velocity Potential and Inviscid Flow Pressure

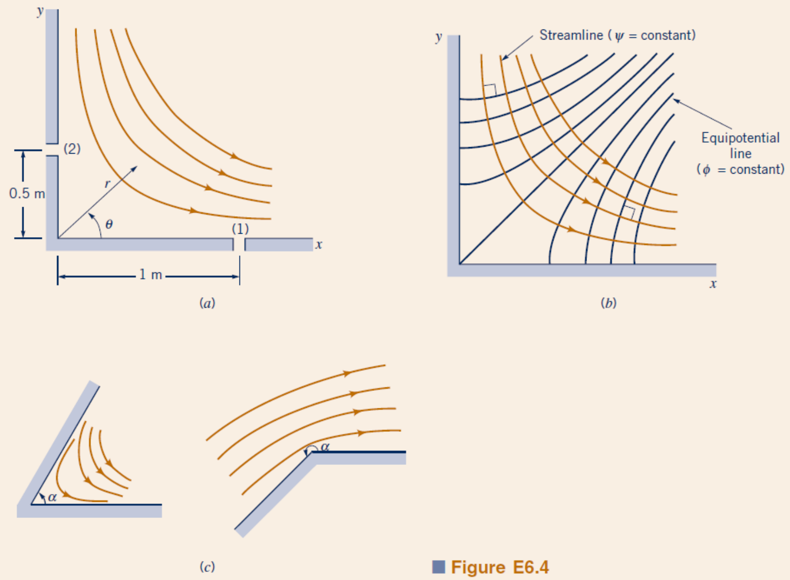
GIVEN The two-dimensional flow of a nonviscous, incompressible fluid in the vicinity of the 90° corner of Fig. E6.4*a* is described by the stream function

$$\psi = 2r^2 \sin 2\theta$$

where ψ has units of m^2/s when r is in meters. Assume the fluid density is $10^3\ kg/m^3$ and the x–y plane is horizontal—that is, there is no difference in elevation between points (1) and (2).

FIND

(a) Determine, if possible, the corresponding velocity potential.
(b) If the pressure at point (1) on the wall is 30 kPa, what is the pressure at point (2)?



■ Figure E6.4

SOLUTION

(a) The radial and tangential velocity components can be obtained from the stream function as (see Eq. 6.42)

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r \cos 2\theta$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -4r \sin 2\theta$$

Since

$$v_r = \frac{\partial \phi}{\partial r}$$

it follows that

$$\frac{\partial \phi}{\partial r} = 4r \cos 2\theta$$

and therefore by integration

$$\phi = 2r^2 \cos 2\theta + f_1(\theta) \qquad (1)$$

where $f_1(\theta)$ is an arbitrary function of θ. Similarly

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -4r \sin 2\theta$$

and integration yields

$$\phi = 2r^2 \cos 2\theta + f_2(r) \qquad (2)$$

where $f_2(r)$ is an arbitrary function of r. To satisfy both Eqs. 1 and 2, the velocity potential must have the form

$$\phi = 2r^2 \cos 2\theta + C \qquad \textbf{(Ans)}$$

where C is an arbitrary constant. As is the case for stream functions, the specific value of C is not important, and it is customary to let $C = 0$ so that the velocity potential for this corner flow is

$$\phi = 2r^2 \cos 2\theta \qquad \textbf{(Ans)}$$

COMMENT In the statement of this problem, it was implied by the wording "if possible" that we might not be able to find a corresponding velocity potential. The reason for this concern is that we can always define a stream function for two-dimensional flow, but the flow must be *irrotational* if there is a corresponding velocity potential. Thus, the fact that we were able to determine a velocity potential means that the flow is irrotational. Several streamlines and lines of constant ϕ are plotted in Fig. E6.4*b*. These two sets of lines are *orthogonal*. The reason why streamlines and lines of constant ϕ are always orthogonal is explained in Section 6.5.

(b) Since we have an irrotational flow of a nonviscous, incompressible fluid, the Bernoulli equation can be applied between any two points. Thus, between points (1) and (2) with no elevation change

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

or

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) \qquad (3)$$

Since

$$V^2 = v_r^2 + v_\theta^2$$

it follows that for any point within the flow field

$$\begin{aligned} V^2 &= (4r \cos 2\theta)^2 + (-4r \sin 2\theta)^2 \\ &= 16r^2(\cos^2 2\theta + \sin^2 2\theta) \\ &= 16r^2 \end{aligned}$$

This result indicates that the square of the velocity at any point depends only on the radial distance, r, to the point. Note that the constant, 16, has units of s^{-2}. Thus,

$$V_1^2 = (16\ s^{-2})(1\ m)^2 = 16\ m^2/s^2$$

and

$$V_2^2 = (16\ s^{-2})(0.5\ m)^2 = 4\ m^2/s^2$$

Substitution of these velocities into Eq. 3 gives

$$\begin{aligned} p_2 &= 30 \times 10^3\ N/m^2 + \frac{10^3\ kg/m^3}{2}(16\ m^2/s^2 - 4\ m^2/s^2) \\ &= 36\ kPa \end{aligned} \qquad \textbf{(Ans)}$$

COMMENT The stream function used in this example could also be expressed in Cartesian coordinates as

$$\psi = 2r^2 \sin 2\theta = 4r^2 \sin\theta \cos\theta$$

or

$$\psi = 4xy$$

since $x = r\cos\theta$ and $y = r\sin\theta$. However, in the cylindrical polar form the results can be generalized to describe flow in the vicinity of a corner of angle α (see Fig. E6.4*c*) with the equations

$$\psi = Ar^{\pi/\alpha} \sin\frac{\pi\theta}{\alpha}$$

and

$$\phi = Ar^{\pi/\alpha} \cos\frac{\pi\theta}{\alpha}$$

where A is a constant.

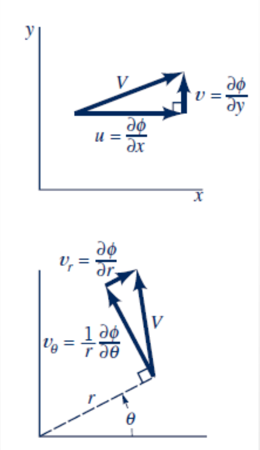
Some Basic, Plane Potential Flows

Laplace equation ➔ Linear PDE (superposition)

2D potential flows

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad \text{or} \qquad v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{or} \qquad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$



Irrotationality condition

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \;\Rightarrow\; \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Stream function

$$\left.\frac{dy}{dx}\right|_{\text{along } \psi = \text{constant}} = \frac{v}{u}$$

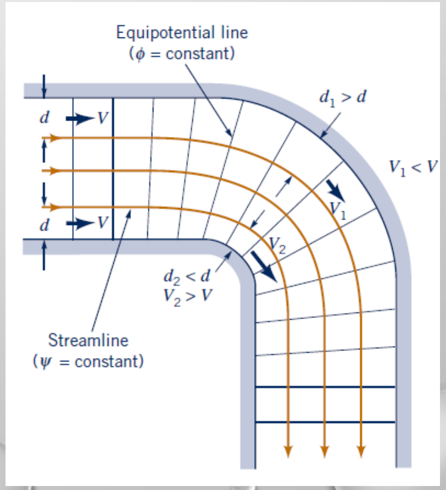
Potential function

Equipotential lines

$$\left.\frac{dy}{dx}\right|_{\text{along } \phi = \text{constant}} = -\frac{u}{v}$$

Lines of constant φ are orthogonal to lines of constant ψ

Flow net



Uniform Flow

$$\frac{\partial \phi}{\partial x} = U \qquad \frac{\partial \phi}{\partial y} = 0$$

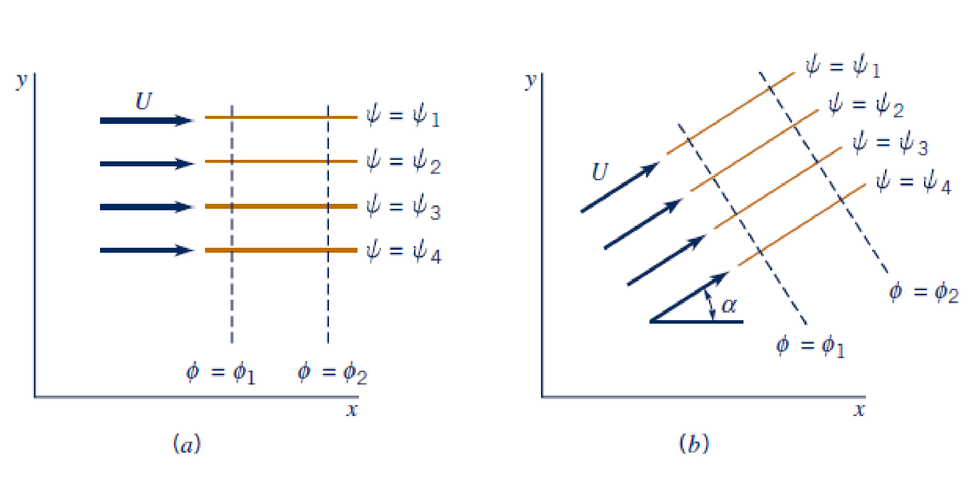
$$\phi = Ux$$

$$\frac{\partial \psi}{\partial y} = U \qquad \frac{\partial \psi}{\partial x} = 0$$

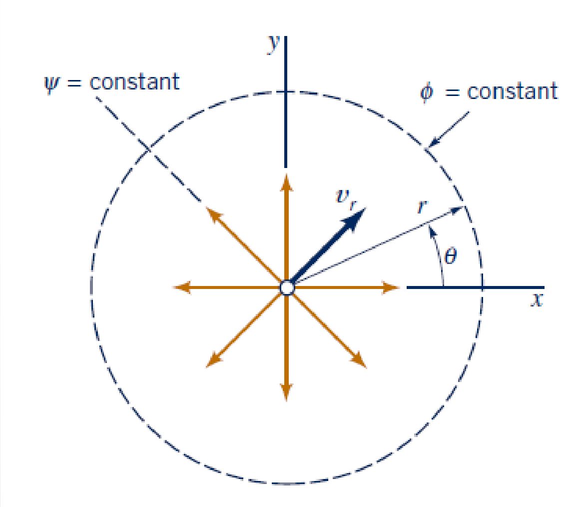
$$\psi = Uy$$

$$\phi = U(x \cos \alpha + y \sin \alpha)$$

$$\psi = U(y \cos \alpha - x \sin \alpha)$$



Source and Sink



$$(2\pi r)v_r = m$$

$$v_r = \frac{m}{2\pi r}$$

$$v_\theta = 0$$

$$\frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

$$\phi = \frac{m}{2\pi} \ln r$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \qquad v_\theta = -\frac{\partial \psi}{\partial r} = 0$$

$$\psi = \frac{m}{2\pi}\theta$$

m(+) source / m(-) sink

m : the strength of the source or sink

EXAMPLE 6.5 Potential Flow—Sink

GIVEN A nonviscous, incompressible fluid flows between wedge-shaped walls into a small opening as shown in Fig. E6.5. The velocity potential (in m^2/s) that approximately describes this flow is

$$\phi = -2 \ln r$$

FIND Determine the volume rate of flow (per unit length) into the opening.

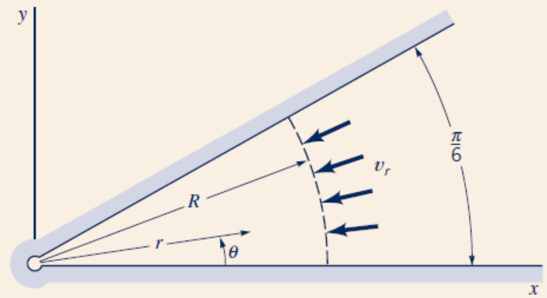


Figure E6.5

SOLUTION

The components of velocity are

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

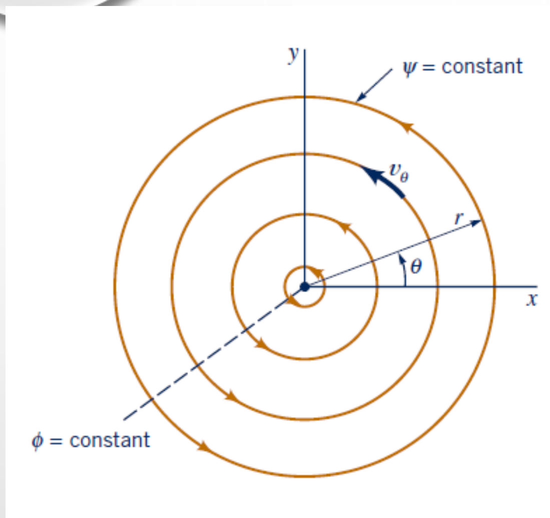
which indicates we have a purely radial flow. The flowrate per unit width, q, crossing the arc of length $R\pi/6$ can thus be obtained by integrating the expression

$$q = \int_0^{\pi/6} v_r R\, d\theta = -\int_0^{\pi/6} \left(\frac{2}{R}\right) R\, d\theta$$

$$= -\frac{\pi}{3} = -1.05 \text{ m}^2/\text{s} \qquad \textbf{(Ans)}$$

COMMENT Note that the radius R is arbitrary since the flowrate crossing any curve between the two walls must be the same. The negative sign indicates that the flow is toward the opening, that is, in the negative radial direction.

Vortex



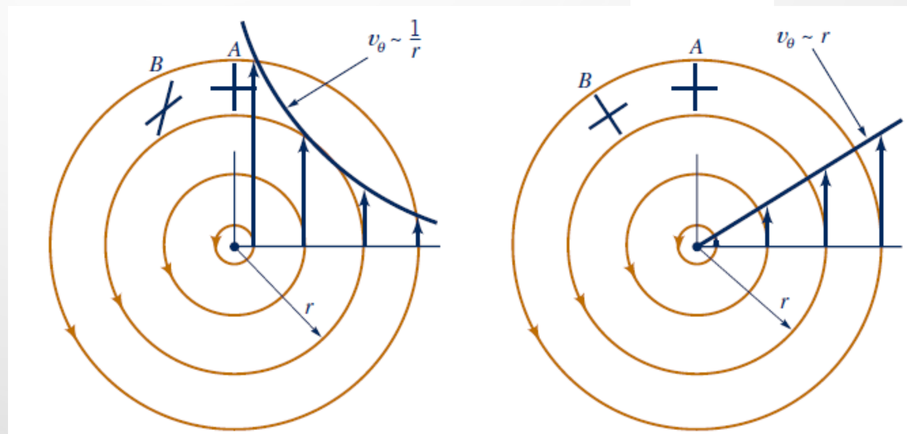
$$\phi = K\theta$$ and $$\psi = -K \ln r$$

$$v_r = 0$$

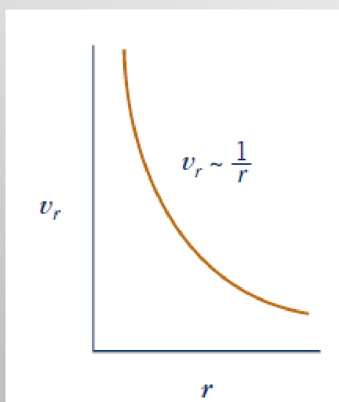
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}$$

Vortex is irrotational?

$$v_\theta = K_1 r$$



Free vortex   Forced vortex

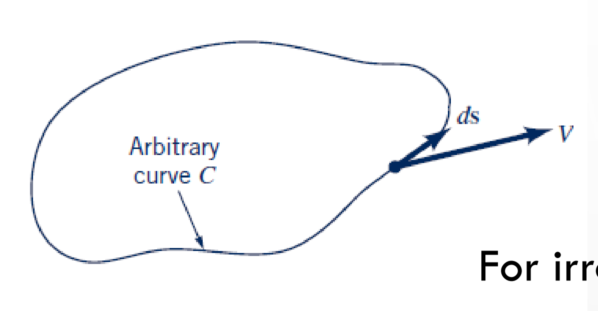


Combined vortex

$$v_\theta = \omega r \qquad r \le r_0$$ and $$v_\theta = \frac{K}{r} \qquad r > r_0$$

Vortex

Circulation



$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$

$$\mathbf{V} = \nabla\phi \qquad \Gamma = \oint_C d\phi = 0$$

For irrotational flow, the circulation will generally be zero

In presence of singularity within the curve (e.g., free vortex)

$$\Gamma = \int_0^{2\pi} \frac{K}{r}(r\,d\theta) = 2\pi K \quad \Rightarrow \quad K = \Gamma/2\pi$$

$$\phi = \frac{\Gamma}{2\pi}\theta$$

and

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

EXAMPLE 6.6 Potential Flow—Free Vortex

GIVEN A liquid drains from a large tank through a small opening as illustrated in Fig. E6.6*a*. A vortex forms whose velocity distribution away from the tank opening can be approximated as that of a free vortex having a velocity potential

$$\phi = \frac{\Gamma}{2\pi}\theta$$

FIND Determine an expression relating the surface shape to the strength of the vortex as specified by the circulation Γ.

SOLUTION

Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

can be written between any two points. If the points are selected at the free surface, $p_1 = p_2 = 0$, so that

$$\frac{V_1^2}{2g} = z_s + \frac{V_2^2}{2g} \qquad \textbf{(1)}$$

where the free surface elevation, z_s, is measured relative to a datum passing through point (1) as shown in Fig. E6.6*b*.

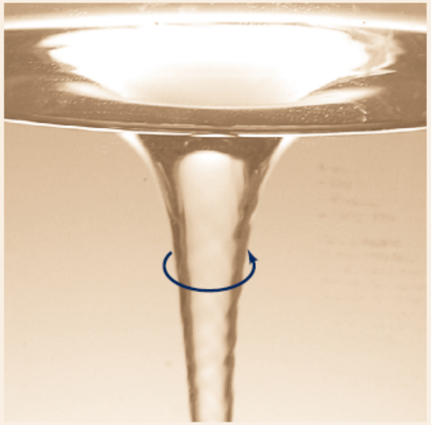
The velocity is given by the equation

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

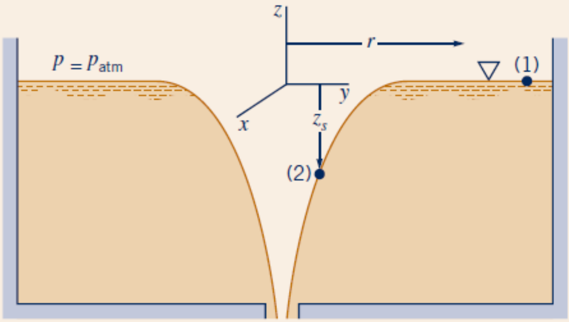
We note that far from the origin at point (1), $V_1 = v_\theta \approx 0$ so that Eq. 1 becomes

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g} \qquad \textbf{(Ans)}$$

which is the desired equation for the surface profile.



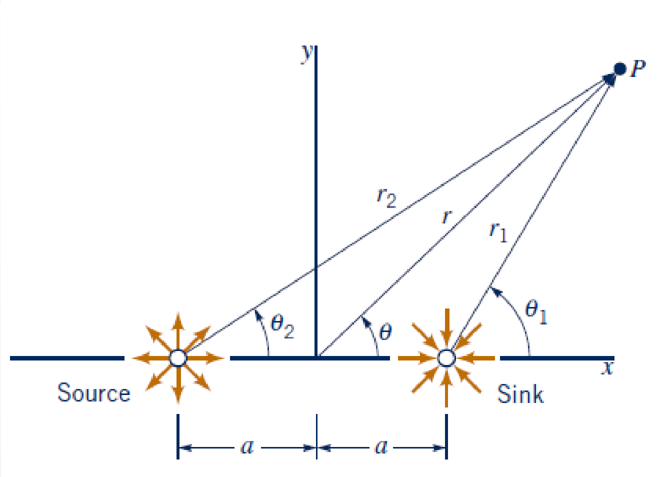
■ Figure E6.6*a*



■ Figure E6.6*b*

COMMENT The negative sign indicates that the surface falls as the origin is approached as shown in Fig. E6.6. This solution is not valid very near the origin since the predicted velocity becomes excessively large as the origin is approached.

Doublet



$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$$ and $$\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$$

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$$ or $$\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

For small distance a

$$\psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)}$$

a → 0 ma/π: strength of doublet

$$\psi = -\frac{K\sin\theta}{r}$$

$$\phi = \frac{K\cos\theta}{r}$$

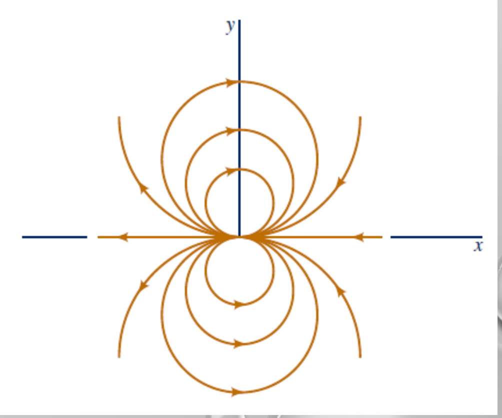


Table 6.1

Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16b)	$\phi = U(x \cos \alpha + y \sin \alpha)$	$\psi = U(y \cos \alpha - x \sin \alpha)$	$u = U \cos \alpha$ $v = U \sin \alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi} \ln r$	$\psi = \dfrac{m}{2\pi} \theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi} \theta$	$\psi = -\dfrac{\Gamma}{2\pi} \ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \dfrac{K \cos \theta}{r}$	$\psi = -\dfrac{K \sin \theta}{r}$	$v_r = -\dfrac{K \cos \theta}{r^2}$ $v_\theta = -\dfrac{K \sin \theta}{r^2}$

[a]Velocity components are related to the velocity potential and stream function through the relationships:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}.$$

Superposition of Basic, Plane Potential Flows

: Various basic velocity potentials and stream functions can be combined to form new potentials and stream functions

: Any streamline in an inviscid flow field can be considered as a solid boundary

➔ **Method of superposition**

Sources in a uniform flow – Half-Body

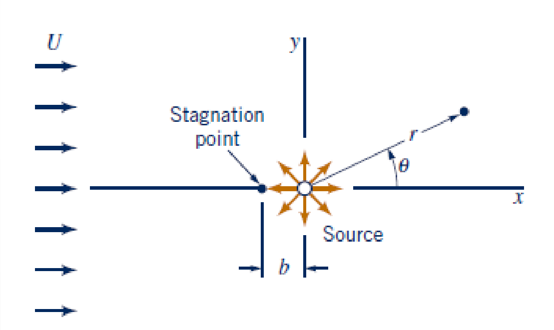
Superposition of a source and a uniform flow

Stream function

$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{source}}$$
$$= Ur \sin\theta + \frac{m}{2\pi}\theta$$

Velocity potential

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$



The stagnation point (x=-b)

$$U = \frac{m}{2\pi b} \quad \text{or} \quad b = \frac{m}{2\pi U}$$

The stream function at the stagnation point (r=b, $\theta=\pi$)

$$\psi_{\text{stagnation}} = \frac{m}{2}$$

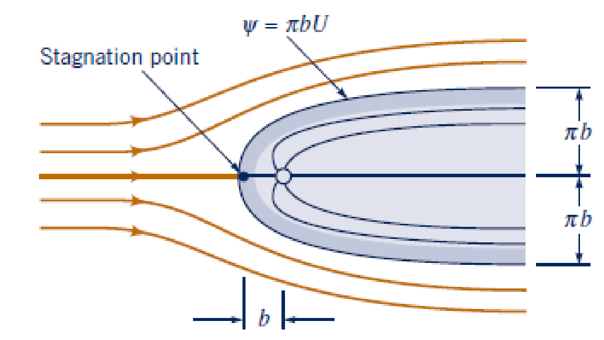
The equation of the streamline passing through the stagnation point

$$\pi b U = Ur\sin\theta + bU\theta \quad \text{or} \quad r = \frac{b(\pi-\theta)}{\sin\theta}$$

Sources in a uniform flow – Half-Body

Superposition of a source and a uniform flow

Half-body



$$r = \frac{b(\pi - \theta)}{\sin\theta}$$ or $$y = b(\pi - \theta)$$

The width of half-body

$$2\pi b = m/U$$

The flow field outside the body

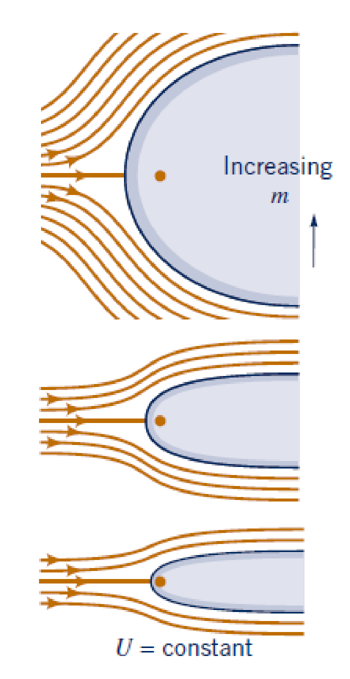
$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta + \frac{m}{2\pi r}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$$

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

The pressure at any point

$$p_0 + \tfrac{1}{2}\rho U^2 = p + \tfrac{1}{2}\rho V^2$$



EXAMPLE 6.7 Potential Flow—Half-body

GIVEN A 64 km/h wind blows toward a hill arising from a plain that can be approximated with the top section of a half-body as illustrated in Fig. E6.7*a*. The height of the hill approaches 60 m as shown. Assume an air density of 1.22 kg/m³.

FIND

(a) What is the magnitude of the air velocity at a point on the hill directly above the origin [point (2)]?

(b) What is the elevation of point (2) above the plain and what is the difference in pressure between point (1) on the plain far from the hill and point (2)?

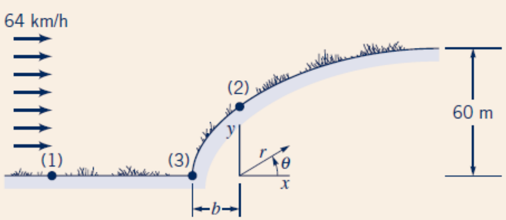


Figure E6.7*a*

SOLUTION

(a) The velocity is given by Eq. 6.101 as

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

At point (2), $\theta = \pi/2$, and since this point is on the surface (Eq. 6.100),

$$r = \frac{b(\pi - \theta)}{\sin\theta} = \frac{\pi b}{2} \qquad (1)$$

Thus,

$$V_2^2 = U^2\left[1 + \frac{b^2}{(\pi b/2)^2}\right] = U^2\left(1 + \frac{4}{\pi^2}\right)$$

and the magnitude of the velocity at (2) for a 64 km/h approaching wind is

$$V_2 = \left(1 + \frac{4}{\pi^2}\right)^{1/2}(64 \text{ km/h}) = 76 \text{ km/h} \qquad \textbf{(Ans)}$$

(b) The elevation at (2) above the plain is given by Eq. 1 as

$$y_2 = \frac{\pi b}{2}$$

Since the height of the hill approaches 60 m and this height is equal to πb, it follows that

$$y_2 = \frac{60 \text{ m}}{2} = 30 \text{ m} \qquad \textbf{(Ans)}$$

From the Bernoulli equation (with the y axis the vertical axis)

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + y_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + y_2$$

so that

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) + \gamma(y_2 - y_1)$$

and with

$$V_1 = (64 \text{ km/h})\left(\frac{1000 \text{ m/km}}{3600 \text{ s/h}}\right) = 17.8 \text{ m/s}$$

and

$$V_2 = (76 \text{ km/h})\left(\frac{1000 \text{ m/km}}{3600 \text{ s/h}}\right) = 21.1 \text{ m/s}$$

it follows that

$$p_1 - p_2 = \frac{(1.22 \text{ kg/m}^3)}{2}[(21.1 \text{ m/s})^2 - (17.8 \text{ m/s})^2] + (1.22 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(30 \text{ m} - 0 \text{ m}) = 440.9 \text{ N/m}^3 = 0.44 \text{ kPa} \qquad \textbf{(Ans)}$$

COMMENTS This result indicates that the pressure on the hill at point (2) is slightly lower than the pressure on the plain at some distance from the base of the hill with a 0.37 kPa difference due to the elevation increase and a 0.07 kPa difference due to the velocity increase.

By repeating the calculations for various values of the upstream wind speed, U, the results shown in Fig. E6.7*b* are obtained. Note that as the wind speed increases, the pressure difference increases from the calm conditions of $p_1 - p_2 = 0.37$ kPa.

The maximum velocity along the hill surface does not occur at point (2) but farther up the hill at $\theta = 63°$. At this point $V_{surface} = 1.26U$. The minimum velocity ($V = 0$) and maximum pressure occur at point (3), the stagnation point.

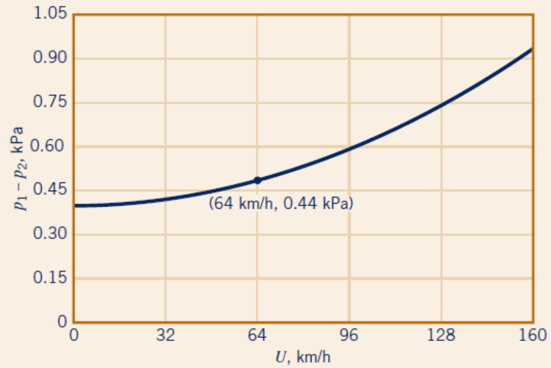


Figure E6.7*b*

Rankine Ovals

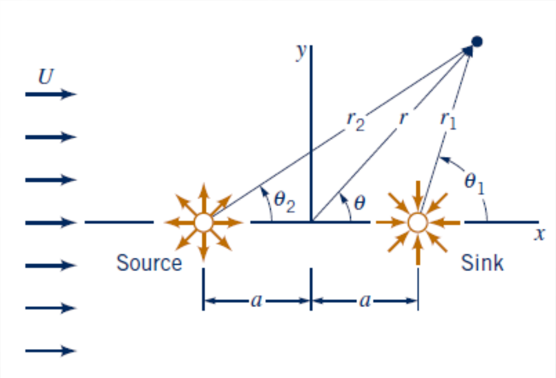
Superposition of source and sink of equal strength and a uniform flow

Stream function

$$\psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2)$$

Velocity potential

$$\phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2)$$



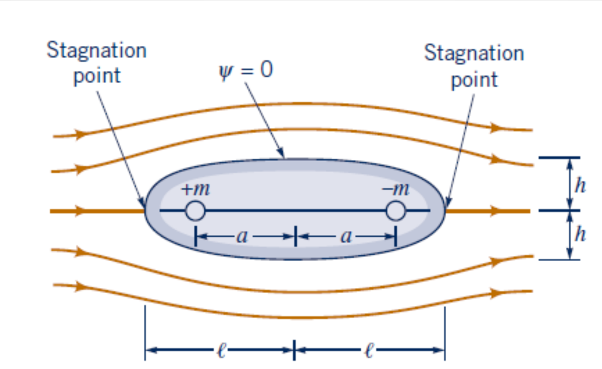
Re-arranged

$$\psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

or

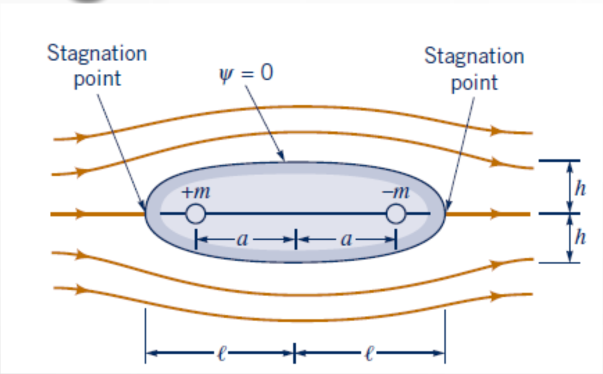
$$\psi = Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2 + y^2 - a^2}\right)$$

$\psi=0$ forms the surface of an oval shape with length 2ℓ and height $2h$



Rankine Ovals

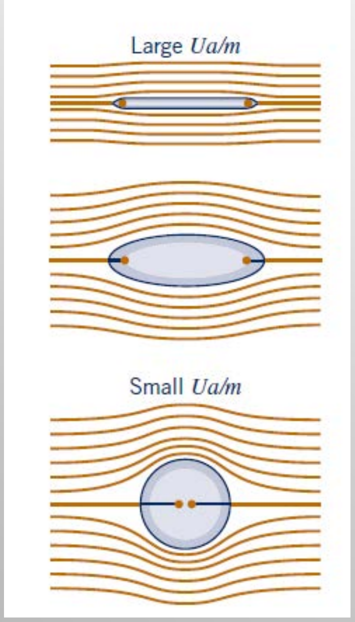
Superposition of source and sink of equal strength and a uniform flow



From stagnation point **V**=0

$$\ell = \left(\frac{ma}{\pi U} + a^2\right)^{1/2}$$

$$\frac{\ell}{a} = \left(\frac{m}{\pi U a} + 1\right)^{1/2}$$



The body half-width h from ψ=0

$$h = \frac{h^2 - a^2}{2a} \tan \frac{2\pi U h}{m}$$

$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right] \tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right]$$

Flow around a circular cylinder

Superposition of a doublet and a uniform flow

<u>Doublet</u>: the distance between the source-sink pair approaches zero

Stream function

$$\psi = Ur\sin\theta - \frac{K\sin\theta}{r}$$

Velocity potential

$$\phi = Ur\cos\theta + \frac{K\cos\theta}{r}$$

A circular cylinder: ψ=constant at $r=a$

$$\psi = \left(U - \frac{K}{r^2}\right) r\sin\theta \quad \Longrightarrow \quad U - \frac{K}{a^2} = 0$$

$\psi=0$ at $r=a$

The stream function for flow around a circular cylinder

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

The velocity potential for flow around a circular cylinder

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$

Flow around a circular cylinder

Superposition of a doublet and a uniform flow

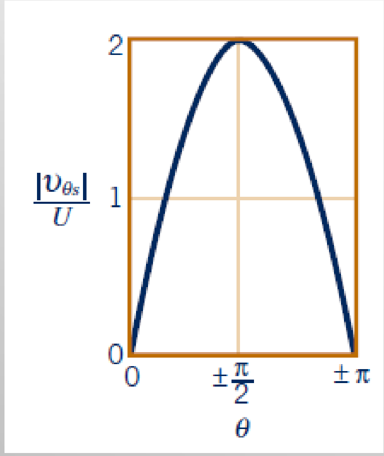
The velocity component

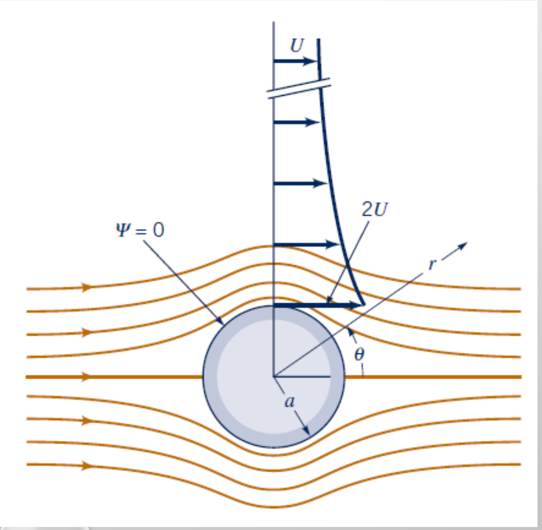
$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

On the surface of the cylinder (r=a)

$$v_r = 0 \qquad v_{\theta s} = -2U\sin\theta$$





Flow around a circular cylinder

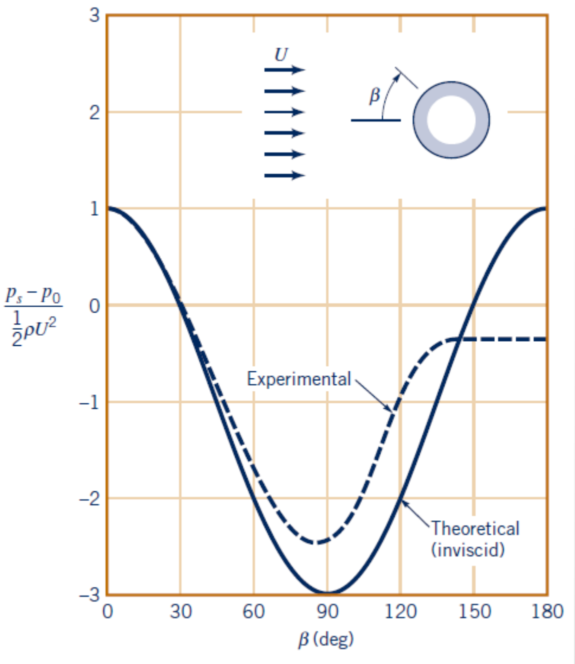
Superposition of a doublet and a uniform flow

The pressure distribution on the surface

$$p_0 + \tfrac{1}{2}\rho U^2 = p_s + \tfrac{1}{2}\rho v_{\theta s}^2$$

$$v_{\theta s} = -2U \sin\theta$$

$$p_s = p_0 + \tfrac{1}{2}\rho U^2(1 - 4$$

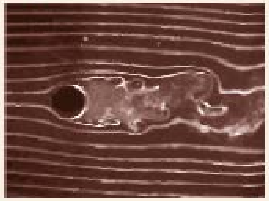


The resultant force (per unit length)

drag $$F_x = -\int_0^{2\pi} p_s \cos\theta \, a \, d\theta$$

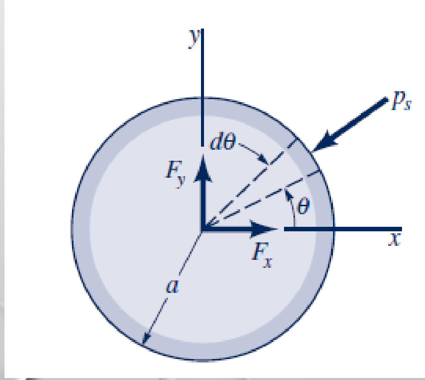
lift $$F_y = -\int_0^{2\pi} p_s \sin\theta \, a \, d\theta$$

V6.8 Circular cylinder with separation



F_x=0 d'Alembert's paradox

F_y=0



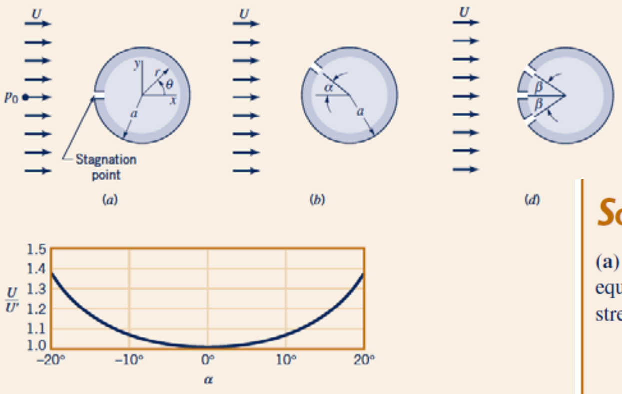
EXAMPLE 6.8 Potential Flow—Cylinder

GIVEN When a circular cylinder is placed in a uniform stream, a stagnation point is created on the cylinder as is shown in Fig. E6.8*a*. If a small hole is located at this point, the stagnation pressure, p_{stag}, can be measured and used to determine the approach velocity, U.

FIND

(a) Show how p_{stag} and U are related.

(b) If the cylinder is misaligned by an angle α (Fig. E6.8*b*), but the measured pressure is still interpreted as the stagnation pressure, determine an expression for the ratio of the true velocity, U, to the predicted velocity, U'. Plot this ratio as a function of α for the range $-20° \leq \alpha \leq 20°$.



■ Figure E6.8

SOLUTION

(a) The velocity at the stagnation point is zero, so the Bernoulli equation written between a point on the stagnation streamline upstream from the cylinder and the stagnation point gives

$$\frac{p_0}{\gamma} + \frac{U^2}{2g} = \frac{p_{stag}}{\gamma}$$

Thus,

$$U = \left[\frac{2}{\rho}(p_{stag} - p_0)\right]^{1/2} \qquad \textbf{(Ans)}$$

COMMENT A measurement of the difference between the pressure at the stagnation point and the upstream pressure can be used to measure the approach velocity. This is, of course, the same result that was obtained in Section 3.5 for Pitot-static tubes.

(b) If the direction of the fluid approaching the cylinder is not known precisely, it is possible that the cylinder is misaligned by some angle, α. In this instance the pressure actually measured, p_α, will be different from the stagnation pressure, but if the misalignment is not recognized the predicted approach velocity, U', would still be calculated as

$$U' = \left[\frac{2}{\rho}(p_\alpha - p_0)\right]^{1/2}$$

Thus,

$$\frac{U(\text{true})}{U'(\text{predicted})} = \left(\frac{p_{stag} - p_0}{p_\alpha - p_0}\right)^{1/2} \qquad \textbf{(1)}$$

The velocity on the surface of the cylinder, v_θ, where $r = a$, is obtained from Eq. 6.115 as

$$v_\theta = -2U \sin\theta$$

If we now write the Bernoulli equation between a point upstream of the cylinder and the point on the cylinder where $r = a$, $\theta = \alpha$, it follows that

$$p_0 + \tfrac{1}{2}\rho U^2 = p_\alpha + \tfrac{1}{2}\rho(-2U\sin\alpha)^2$$

and, therefore,

$$p_\alpha - p_0 = \tfrac{1}{2}\rho U^2(1 - 4\sin^2\alpha) \qquad \textbf{(2)}$$

Since $p_{stag} - p_0 = \frac{1}{2}\rho U^2$ it follows from Eqs. 1 and 2 that

$$\frac{U(\text{true})}{U'(\text{predicted})} = (1 - 4\sin^2\alpha)^{-1/2} \qquad \textbf{(Ans)}$$

This velocity ratio is plotted as a function of the misalignment angle α in Fig. E6.8*c*.

COMMENT It is clear from these results that significant errors can arise if the stagnation pressure tap is not aligned with the stagnation streamline. As is discussed in Section 3.5, if two additional, symmetrically located holes are drilled on the cylinder, as are illustrated in Fig. E6.8*d*, the correct orientation of the cylinder can be determined. The cylinder is rotated until the pressures in the two symmetrically placed holes are equal, thus indicating that the center hole coincides with the stagnation streamline. For $\beta = 30°$ the pressure at the two holes theoretically corresponds to the upstream pressure, p_0. With this orientation a measurement of the difference in pressure between the center hole and the side holes can be used to determine U.

Flow around a circular cylinder

Flow around a cylinder + a free vortex

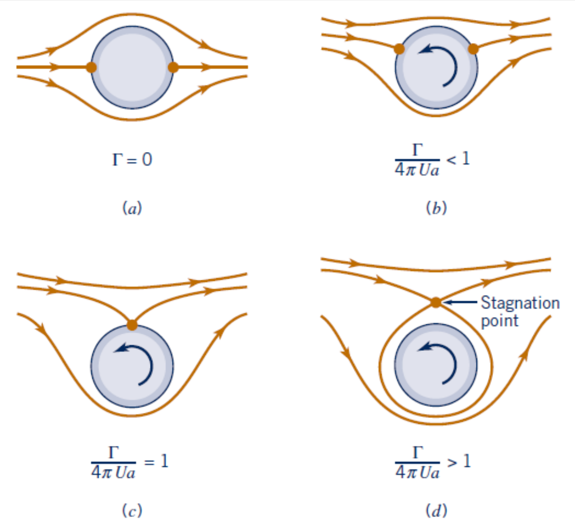
Stream function $$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r$$

r=a still streamline

Velocity potential $$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

The tangential velocity $$v_{\theta s} = -\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = -2U\sin\theta + \frac{\Gamma}{2\pi a}$$

The location of stagnation points ($v_\theta=0$) $$\sin\theta_{stag} = \frac{\Gamma}{4\pi Ua}$$



Flow around a circular cylinder

Flow around a cylinder + a free vortex

The surface pressure p_s from the Bernoulli equation

$$p_0 + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$

$$p_s = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi a U} - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\right)$$

drag $F_x = 0$

lift $F_y = -\rho U \Gamma$

Magnus effect

Generalized equation

→ Kutta-Joukowski law

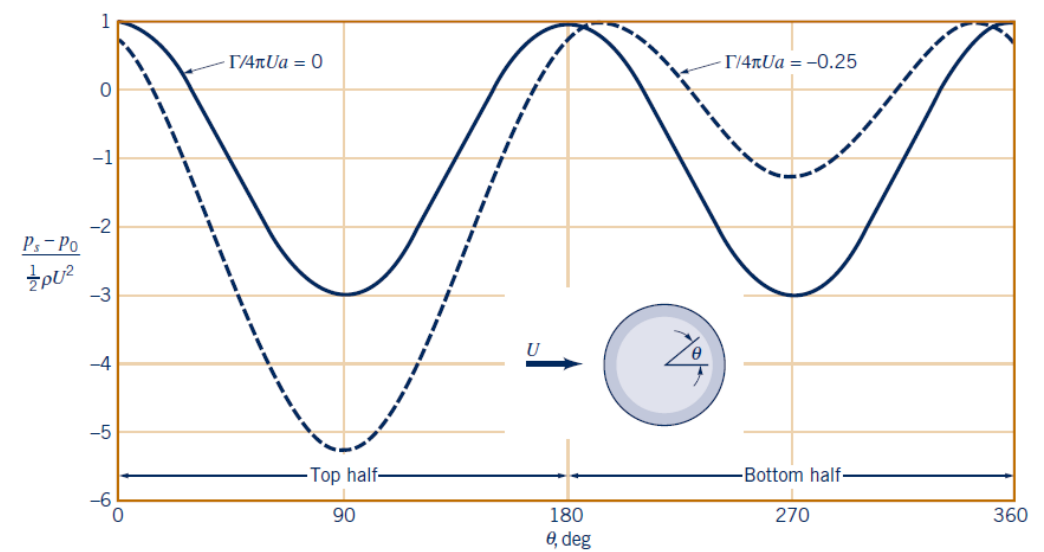


Figure 6.30 Pressure distribution on a circular cylinder with and without rotation.

Viscous flow

Stress-Deformation Relationships

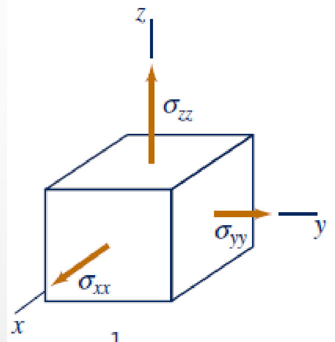
For incompressible, Newtonian fluids

Normal stress

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$



$$p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

Shearing stress

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

For Newtonian fluids, the stresses are linearly related to the rate of deformation

The Navier-Stokes Equation

For rectangular coordinates

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

For cylindrical coordinates

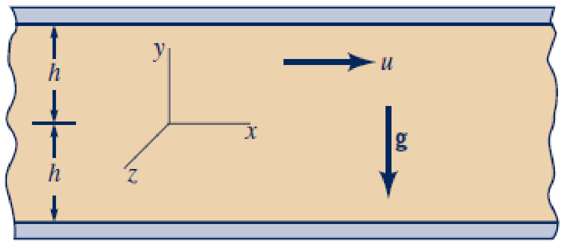
$$\begin{aligned}\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)\\ =-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]\end{aligned}$$

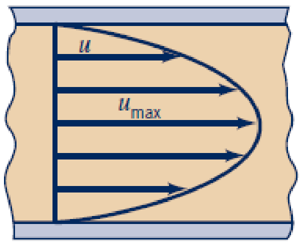
$$\begin{aligned}\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)\\ =-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]\end{aligned}$$

$$\begin{aligned}\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)\\ =-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]\end{aligned}$$

Some Simple Solutions for Laminar, Viscous, Incompressible Fluids

Steady, Laminar Flow between Fixed Parallel Plates





v=0, w=0

$$\partial u/\partial x = 0$$

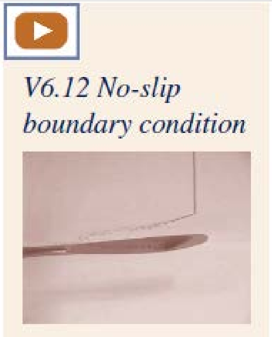
$$\partial u/\partial t = 0$$

$$u = u(y)$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \;\Rightarrow\; \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \;\Rightarrow\; u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2$$

$$0 = -\frac{\partial p}{\partial y} - \rho g$$

$$0 = -\frac{\partial p}{\partial z} \;\Rightarrow\; p = -\rho g y + f_1(x)$$



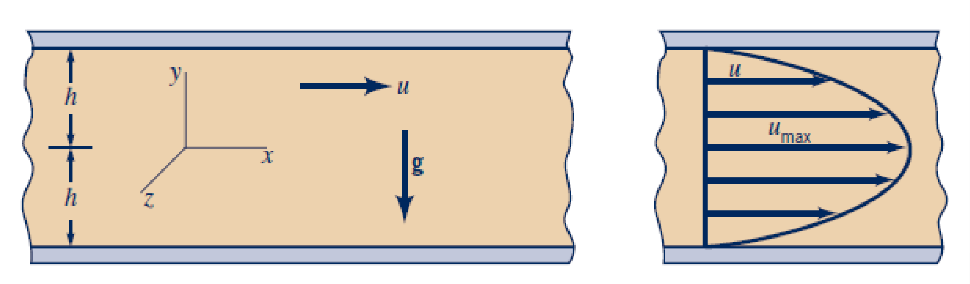
V6.12 No-slip boundary condition

Boundary conditions $u = 0 \text{ for } y = \pm h$

$$c_1 = 0$$

$$c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2 \;\Rightarrow\; u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)$$

Steady, Laminar Flow between Fixed Parallel Plates



The volume flow rate q, passing between the plates

$$q = \int_{-h}^{h} u\, dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)\, dy$$

$$q = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

or

$$q = \frac{2h^3 \Delta p}{3\mu \ell}$$

The mean velocity V=q/2h

$$V = \frac{h^2 \Delta p}{3\mu \ell}$$

The maximum velocity

$$u_{\max} = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)$$

or

$$u_{\max} = \tfrac{3}{2} V$$

The pressure

$$p = -\rho g y + f_1(x)$$

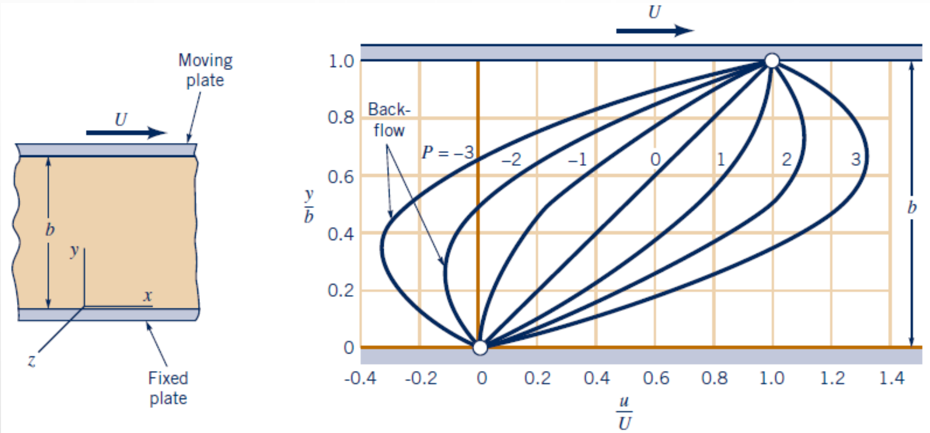
$$f_1(x) = \left(\frac{\partial p}{\partial x}\right) x + p_0$$

$$p = -\rho g y + \left(\frac{\partial p}{\partial x}\right) x + p_0$$

p_0: reference pressure at x=y=0

Couette flow

Moving plate



Boundary conditions

u=0 at y=0

u=U at y=b

$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by)$$

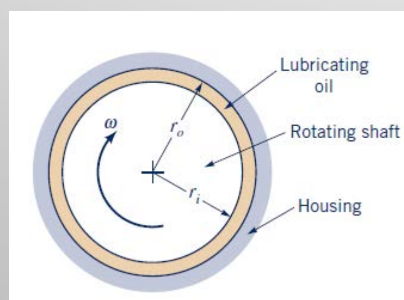
or

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$

Dimensionless parameter $P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$

For $\partial p/\partial x = 0$ $\quad u = U\frac{y}{b}$

Example: the flow between closely spaced concentric cylinders



$$U = r_i \omega$$

$$b = r_o - r_i$$

$$\tau = \mu r_i \omega/(r_o - r_i)$$

EXAMPLE 6.9 Plane Couette Flo

GIVEN A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0, as illustrated in Fig. E6.9*a*. Because of viscous forces the belt picks up a film of fluid of thickness h. Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed.

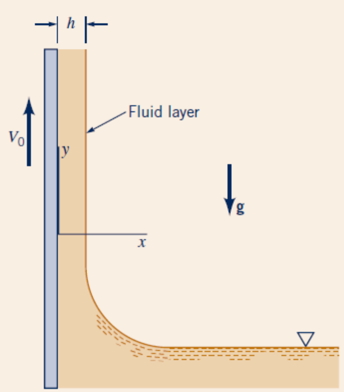
FIND Use the Navier–Stokes equations to determine an expression for the average velocity of the fluid film as it is dragged up the belt.

SOLUTION

Since the flow is assumed to be fully developed, the only velocity component is in the y direction (the v component) so that $u = w = 0$. It follows from the continuity equation that $\partial v/\partial y = 0$, and for steady flow $\partial v/\partial t = 0$, so that $v = v(x)$. Under these conditions the Navier–Stokes equations for the x direction (Eq. 6.127*a*) and the z direction (perpendicular to the paper) (Eq. 6.127*c*) simply reduce to

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial z} = 0$$

w



■ **Figure E6.9*a***

This result indicates that the pressure does not vary over a horizontal plane, and since the pressure on the surface of the film ($x = h$) is atmospheric, the pressure throughout the film must be atmospheric (or zero gage pressure). The equation of motion in the y direction (Eq. 6.127*b*) thus reduces to

$$0 = -\rho g + \mu \frac{d^2 v}{dx^2}$$

or

$$\frac{d^2 v}{dx^2} = \frac{\gamma}{\mu} \qquad \textbf{(1)}$$

Integration of Eq. 1 yields

$$\frac{dv}{dx} = \frac{\gamma}{\mu}x + c_1 \qquad \textbf{(2)}$$

On the film surface ($x = h$) we assume the shearing stress is zero—that is, the drag of the air on the film is negligible. The shearing stress at the free surface (or any interior parallel surface) is designated as τ_{xy}, where from Eq. 6.125*d*

$$\tau_{xy} = \mu\left(\frac{dv}{dx}\right)$$

Thus, if $\tau_{xy} = 0$ at $x = h$, it follows from Eq. 2 that

$$c_1 = -\frac{\gamma h}{\mu}$$

A second integration of Eq. 2 gives the velocity distribution in the film as

$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + c_2$$

At the belt ($x = 0$) the fluid velocity must match the belt velocity, V_0, so that

$$c_2 = V_0$$

and the velocity distribution is therefore

$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0 \qquad \textbf{(3)}$$

With the velocity distribution known we can determine the flowrate per unit width, q, from the relationship

$$q = \int_0^h v\,dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0\right)dx$$

and thus

$$q = V_0 h - \frac{\gamma h^3}{3\mu}$$

The average film velocity, V (where $q = Vh$), is therefore

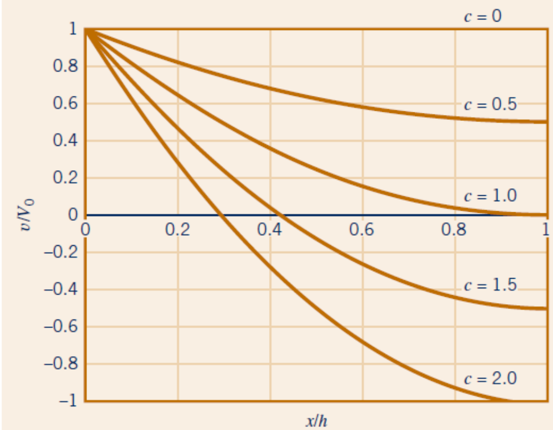
$$V = V_0 - \frac{\gamma h^2}{3\mu} \qquad \textbf{(Ans)}$$

COMMENT Equation (3) can be written in dimensionless form as

$$\frac{v}{V_0} = c\left(\frac{x}{h}\right)^2 - 2c\left(\frac{x}{h}\right) + 1$$

where $c = \gamma h^2/2\mu V_0$. This velocity profile is shown in Fig. E6.9*b*. Note that even though the belt is moving upward, for $c > 1$ (e.g., for fluids with small enough viscosity or with a small enough belt speed) there are portions of the fluid that flow downward (as indicated by $v/V_0 < 0$).

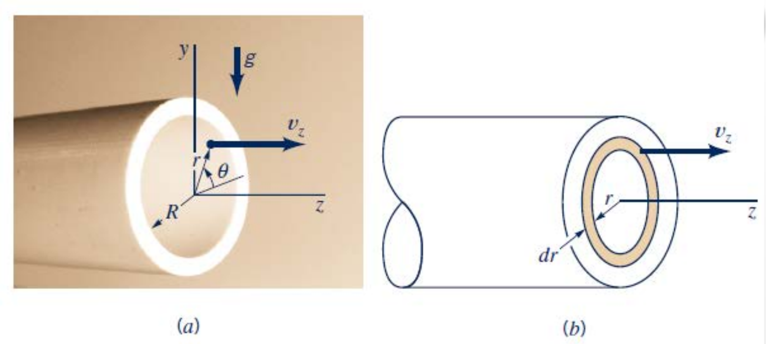
It is interesting to note from this result that there will be a net upward flow of liquid (positive V) only if $V_0 > \gamma h^2/3\mu$. It takes a relatively large belt speed to lift a small viscosity fluid.



■ **Figure E6.9*b***

Steady, Laminar Flow in Circular Tubes

Poiseuille flow or Hagen-Poiseuille flow



Assume that the flow is parallel to the walls

$$v_r = 0, \quad v_\theta = 0, \quad \partial v_z / \partial z = 0$$

Axisymmetric flow $\quad v_z = v_z(r)$

$$g_r = -g \sin\theta \qquad g_\theta = -g\cos\theta$$

$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r}$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$p = -\rho g (r\sin\theta) + f_1(z) \qquad p = -\rho g y + f_1(z)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

$$\partial p / \partial z = \text{constant}$$

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2$$

Steady, Laminar Flow in Circular Tubes

Boundary conditions

$v_z(r=0)$, finite / $v_z(r=R)=0$

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2)$$

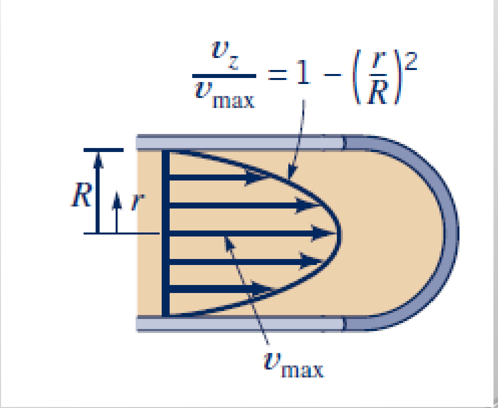
The volume rate of flow Q

$$dQ = v_z(2\pi r)\,dr \qquad Q = 2\pi\int_0^R v_z r\,dr \qquad Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right)$$

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial z} \qquad Q = \frac{\pi R^4 \Delta p}{8\mu\ell}$$

The mean velocity V=Q/A

$$V = \frac{R^2\Delta p}{8\mu\ell}$$

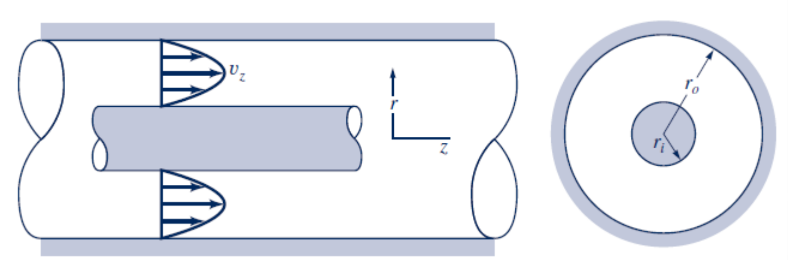


The maximum velocity

$$v_{max} = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2\Delta p}{4\mu\ell} \qquad v_{max} = 2V$$

$$\frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

Steady, Axial, Laminar Flow in an Annulus



Boundary conditions

$v_z(r=r_i)=0$ / $v_z(r=r_o)=0$

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)}\ln\frac{r}{r_o}\right]$$

The volume rate of flow

$$Q = \int_{r_i}^{r_o} v_z(2\pi r)\,dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

$$Q = \frac{\pi\Delta p}{8\mu\ell}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

The maximum velocity ($\partial v_z / \partial r = 0 \quad at \quad r = r_m$)

$$r_m = \left[\frac{r_o^2 - r_i^2}{2\ln(r_o/r_i)}\right]^{1/2}$$

Nearer the inner cylinder

Hydraulic diameter – characteristic length for Reynolds number

$$D_h = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

$$D_h = \frac{4\pi(r_o^2 - r_i^2)}{2\pi(r_o + r_i)} = 2(r_o - r_i)$$

For an annulus

EXAMPLE 6.10 Laminar Flow in an Annulus

GIVEN A viscous liquid ($\rho = 1.18 \times 10^3$ kg/m^3; $\mu = 0.0045$ N · s/m^2) flows at a rate of 12 ml/s through a horizontal, 4 mm diameter tube.

FIND (a) Determine the pressure drop along a 1 m length of the tube which is far from the tube entrance so that the only component of velocity is parallel to the tube axis. (b) If a 2 mm diameter rod is placed in the 4 mm diameter tube to form a symmetric annulus, what is the pressure drop along a 1 m length if the flowrate remains the same as in part (a)?

SOLUTION

(a) We first calculate the Reynolds number, Re, to determine whether or not the flow is laminar. With the diameter $D = 4$ mm $= 0.004$ m, the mean velocity is

$$V = \frac{Q}{(\pi/4)D^2} = \frac{(12 \text{ ml/s})(10^{-6} \text{ m}^3/\text{ml})}{(\pi/4)(0.004 \text{ m})^2} = 0.955 \text{ m/s}$$

and, therefore,

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(1.18 \times 10^3 \text{ kg/m}^3)(0.955 \text{ m/s})(0.004 \text{ m})}{0.0045 \text{ N} \cdot \text{s/m}^2} = 1000$$

Since the Reynolds number is well below the critical value of 2100, we can safely assume that the flow is laminar. Thus, we can apply Eq. 6.151, which gives for the pressure drop

$$\Delta p = \frac{8\mu \ell Q}{\pi R^4} = \frac{8(0.0045 \text{ N} \cdot \text{s/m}^2)(1 \text{ m})(12 \times 10^{-6} \text{ m}^3/\text{s})}{\pi(0.002 \text{ m})^4} = 8.59 \text{ kPa} \qquad \textbf{(Ans)}$$

(b) For flow in the annulus with an outer radius $r_o = 0.002$ m and an inner radius $r_i = 0.001$ m, the mean velocity is

$$V = \frac{Q}{\pi(r_o^2 - r_i^2)} = \frac{12 \times 10^{-6} \text{ m}^3/\text{s}}{(\pi)[(0.002 \text{ m})^2 - (0.001 \text{ m})^2]} = 1.27 \text{ m/s}$$

and the Reynolds number [based on the hydraulic diameter, $D_h = 2(r_o - r_i) = 2(0.002 \text{ m} - 0.001 \text{ m}) = 0.002$ m] is

$$\text{Re} = \frac{\rho D_h V}{\mu} = \frac{(1.18 \times 10^3 \text{ kg/m}^3)\,(0.002 \text{ m})\,(1.27 \text{ m/s})}{0.0045 \text{ N} \cdot \text{s/m}^2} = 666$$

This value is also well below 2100, so the flow in the annulus should also be laminar. From Eq. 6.156,

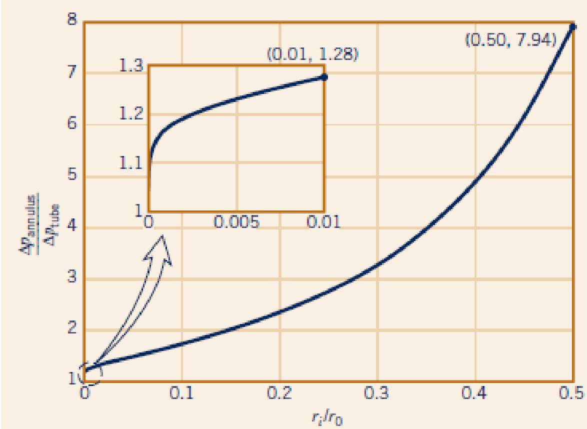
$$\Delta p = \frac{8\mu \ell Q}{\pi}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]^{-1}$$

so that

$$\Delta p = \frac{8(0.0045 \text{ N} \cdot \text{s/m}^2)(1 \text{ m})(12 \times 10^{-6} \text{ m}^3/\text{s})}{\pi} \times \left\{(0.002 \text{ m})^4 - (0.001 \text{ m})^4 - \frac{[(0.002 \text{ m})^2 - (0.001 \text{ m})^2]^2}{\ln(0.002 \text{ m}/0.001 \text{ m})}\right\}^{-1} = 68.2 \text{ kPa} \qquad \textbf{(Ans)}$$

COMMENTS The pressure drop in the annulus is much larger than that of the tube. This is not a surprising result, since to maintain the same flow in the annulus as that in the open tube, the average velocity must be larger (the cross-sectional area is smaller) and the pressure difference along the annulus must overcome the shearing stresses that develop along both an inner and an outer wall.

By repeating the calculations for various radius ratios, r_i/r_o, the results shown in Fig. E6.10 are obtained. It is seen that the pressure drop ratio, $\Delta p_{\text{annulus}}/\Delta p_{\text{tube}}$ (i.e., the pressure drop in the annulus compared to that in a tube with a radius equal to the outer radius of the annulus, r_o), is a strong function of the radius ratio. Even an annulus with a very small inner radius will have a pressure drop significantly larger than that of a tube. For example, if the inner radius is only 1/100 of the outer radius, $\Delta p_{\text{annulus}}/\Delta p_{\text{tube}} = 1.28$. As shown in the figure, for larger inner radii, the pressure drop ratio is much larger [i.e., $\Delta p_{\text{annulus}}/\Delta p_{\text{tube}} = 7.94$ for $r_i/r_o = 0.50$ as in part (b) of this example].



■ Figure E6.10